Isothermal Two-Phase Flow

This process ignores temperature effects and the partially saturated sample is treated as an immiscible two-phase system within the voids of the solid skeleton. In the pressure-pressure formulation the primary variables are (1) gas pressure p^g and (2) capillary pressure p^c. In the pressure-saturation formulation the primary variables are non-wetting phase saturation S^{nw} and wetting-phase pressure p^w. In the benchmarks shown here, both fluids are assumed incompressible[1].

1. Mathematical Framework

Mass balance equation

Consider two-phase flow in porous media, e.g liquid (denoted by l) and gas (denoted by g). For each phase in two-phase fluid flow, mass conservation is given by the following equation,

$$\frac{\partial}{\partial t}(nS^g\rho_k^g + nS^l\rho_k^l) + \nabla \cdot (\mathrm{J}_k^g + \mathrm{J}_k^l) = Q_k \tag{1}$$

where S is saturation, ρ stands for phase density, n is the porosity, $\mathbf{J}$ is total flux. The subscript k in equation (1) denotes the component, e.g air ($k = a$) or water ($k = w$). For any phase $\gamma = (g, l)$, an advection vector $\mathbf{J}_{Ak}^\gamma$ and a diffusion vector $\mathbf{J}_{Dk}^\gamma$ comprise the total flux, i.e.,

$$\mathbf{J}_k^\gamma = \mathbf{J}_{Ak}^\gamma + \mathbf{J}_{Dk}^\gamma \tag{2}$$

According to Darcy's law, the advective part of the total flux may be written as

$$\mathbf{J}_{Ak}^\gamma = -\rho_k^\gamma \frac{\mathbf{k}k_{rel}^\gamma}{\mu^\gamma}(\nabla p^\gamma - \rho^\gamma \mathbf{g}), \tag{3}$$

where $\mathbf{k}$ is the intrinsic permeability, k_{rel}^γ is the relative permeability of the phase γ, and μ^γ is the viscosity.

The diffusive part of the total flux is given by Fick's law

$$\mathbf{J}_{Dk}^\gamma = -nS^\gamma\rho^\gamma\mathbb{D}_k^\gamma\nabla\left(\frac{\rho_k^\gamma}{\rho^\gamma}\right), \tag{4}$$

where $\mathbb{D}$ is the diffusion coefficient tensor. Since $\rho^\gamma = \rho_a^\gamma + \rho_w^\gamma$, we have

$$\mathbf{J}_{Dw}^\gamma + \mathbf{J}_{Da}^\gamma = \mathbf{0} \tag{5}$$

under the assumption $\mathbb{D}_a^\gamma = \mathbb{D}_w^\gamma$.

Consider a water-air mixture. We expand the mass balance equation (1) with the flux defined in equations (2) . For the water component, the diffusive part of the total flux takes the form

$$\mathbf{J}_{D_w}^l = -nS^l\rho^l\mathbb{D}_w^l\nabla\left(\frac{\rho_w^l}{\rho^l}\right), \quad \mathbf{J}_{D_w}^g = -nS^g\rho^g\mathbb{D}_w^g\nabla\left(\frac{\rho_w^g}{\rho^g}\right). \tag{6}$$

Obviously, $\mathbb{D}_w^l = \mathbf{0}$. Therefore, the mass balance equation for water component can be written as follows

$$\frac{\partial}{\partial t}\left(nS^g\rho_w^g + nS^l\rho_w^l\right) - \nabla\cdot\left[\rho_w^l\frac{\mathbf{k}k_{rel}^l}{\mu^l}\left(\nabla p^l - \rho^l\mathbf{g}\right)\right]$$
$$-\nabla\cdot\left[\rho_w^g\frac{\mathbf{k}k_{rel}^g}{\mu^g}\left(\nabla p^g - \rho^g\mathbf{g}\right)\right] - \nabla\cdot\left[nS^g\rho^g\mathbb{D}_w^g\nabla\left(\frac{\rho_w^g}{\rho^g}\right)\right] = Q_w. \tag{7}$$

Since the capillary pressure p^c is chosen as one of the two unknowns of equation (1) and $S^g = 1 - S^l$, equation (7) becomes

$$n(\rho_w^l - \rho_w^g)\frac{\partial S^l}{\partial t} + (1-S^l)n\frac{\partial\rho_w^g}{\partial t} - \nabla\cdot\left[\rho_w^l\frac{\mathbf{k}k_{rel}^l}{\mu^l}\left(\nabla(p^g - p^c) - \rho^l\mathbf{g}\right)\right]$$
$$-\nabla\cdot\left[\rho_w^g\frac{\mathbf{k}k_{rel}^g}{\mu^g}\left(\nabla p^g - \rho^g\mathbf{g}\right)\right] - \nabla\cdot\left[nS^g\rho^g\mathbb{D}_w^g\nabla\left(\frac{\rho_w^g}{\rho^g}\right)\right] = Q_w. \tag{8}$$

Similar to the previous procedure, the diffusion part of the total flux of air component can be written as

$$\mathbf{J}_{D_a}^l = -nS^l\rho^l\mathbb{D}_a^l\nabla\left(\frac{\rho_a^l}{\rho^l}\right), \quad \mathbf{J}_{D_a}^a = -nS^g\rho^g\mathbb{D}_a^g\nabla\left(\frac{\rho_a^g}{\rho^g}\right). \tag{9}$$

The density shift from air component to liquid ρ_a^l is very small and can be omitted. Therefore, we can assume $\mathbf{J}_{D_a}^l \approx 0$. As a consequence, the mass balance equation for air component is derived:

$$\frac{\partial}{\partial t}\left(nS^g\rho_a^g\right) -$$
$$\nabla\cdot\left[\rho_a^g\frac{\mathbf{k}k_{rel}^g}{\mu^g}\left(\nabla p^g - \rho^g\mathbf{g}\right)\right] - \nabla\cdot\left[nS^g\rho^g\mathbb{D}_a^g\nabla\left(\frac{\rho_a^g}{\rho^g}\right)\right] = Q_a. \tag{10}$$

Expanding the temporary derivative term of equation (1) yields

$$-n\rho_a^g\frac{\partial S^l}{\partial t} + (1-S^l)n\frac{\partial\rho_a^g}{\partial t} -$$
$$\nabla\cdot\left[\rho_a^g\frac{\mathbf{k}k_{rel}^g}{\mu^g}\left(\nabla p^g - \rho^g\mathbf{g}\right)\right] - \nabla\cdot\left[nS^g\rho^g\mathbb{D}_a^g\nabla\left(\frac{\rho_a^g}{\rho^g}\right)\right] = Q_a. \tag{11}$$

The mass balance equations (8) and (11) are exactly the same as described in [2].

1.1. Pressure-pressure (pp) scheme

Based on the description of isothermal two-phase flow above, (8) and (11) can be modified in order to obtain governing equations for isothermal two-phase flow in a porous medium. In this formulation primary variables are gas pressure p^g, and capilary pressure p^c.

The basic equations of the isothermal two-phase flow system are:

$$n\rho_w \frac{\partial S_w}{\partial p_c}\dot{p}_c + \nabla \cdot \left[\rho_w \frac{\mathbf{k}k_{relw}}{\mu_w}\left(-\nabla p^g + \nabla p^c + \rho_w \mathbf{g}\right)\right] = Q_w \tag{12}$$

$$-n\rho^a \frac{\partial S_w}{\partial p_c}\dot{p}_c + n(1 - S_w)\left(\frac{\partial \rho_a}{\partial p^g}\dot{p}^g + \frac{\partial \rho_a}{\partial p_c}\dot{p}_c\right) +$$

$$\nabla \cdot \left[\rho_a \frac{\mathbf{k}k_{rela}}{\mu_a}\left(-\nabla p^g + \rho_a \mathbf{g}\right)\right] = Q_a \tag{13}$$

2. Liakopoulos experiment

Definition

This benchmark is based on an experiment by Liakopoulus[3]and is proposed by Lewis and Schrefler [4](pp 167–174).

Results

The temporal evolution of vertical profiles of primary variables (capillary and gas pressure) are given in Fig. 1. The results agree well with the results generated by OGS5.

Property	Symbol	Value	Unit
Porosity	n	$-$	2.975×10^{-1}
Permeability	κ	m^2	4.5000×10^{-13}
Liquid dynamic viscosity	μ_w	$Pa.s$	1.0000×10^{-3}
Gas dynamic viscosity	μ_a	$Pa.s$	1.8×10^{-5}
Liquid density	ρ_w	$kg.m^{-3}$	1.0000×10^3
Gas density	ρ_a	$kg.m^{-3}$	Ideal Gas Law's
Capillary pressure	p^c	Pa	Experimental Curve
Relative permeability	κ_{relw}	$-$	Experimenta Curve
Relative permeability	κ_{rela}	$-$	Brook-Corey functions

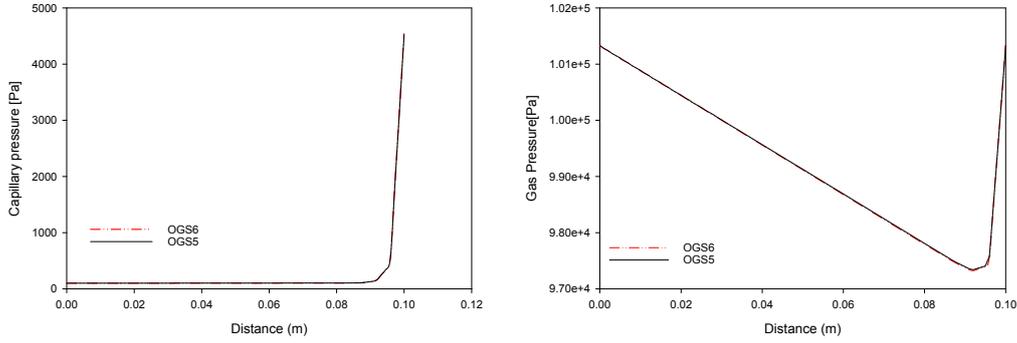Table 1: Material parameters for the Liakopoulos problem.

Figure 1: Vertical profiles of capillary (top) and gas pressures (bottom).

3. McWhorter problem

Problem definition

The test benchmark problem is proposed by McWhorter and Sunada [5]. It is applied to simulate the capillary effects in immiscible two-phase flow transport. The benchmark is formulated as if the instantaneous displacement occurs in one-dimensional horizontal reservoir initially occupied by oil. Solution has been obtain through solving the governing equations 13 by pressure-pressure scheme described in sec (sec.1.1). Here the flow is governed by capillary force when water saturation at the left end of the horizontal column is kept to be one, while the right end is kept to be no flux at all. So for no source term is accounted.
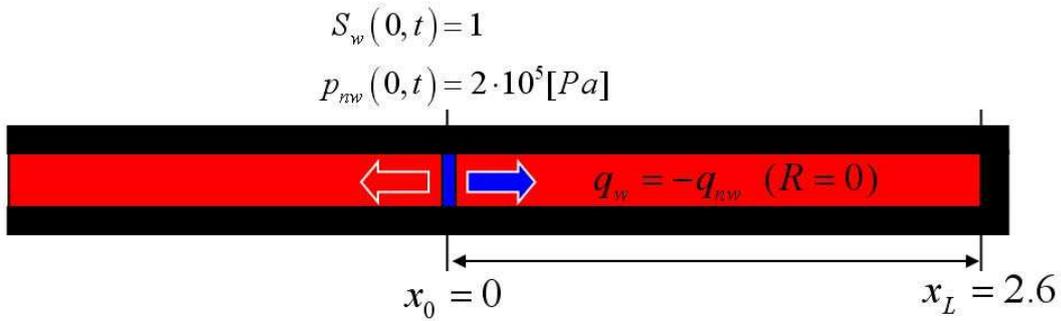

Figure 2: Schematic of the benchmark formulated to test McWhorter and Sunada's analytical solution.

Results

Based on the above discussion OpenGeoSys-6 produces agreeable solution compared with OpenGeoSys-5. Fig. 3 shows water saturation profile, S_w with a fine grid along with $2.6m$ long horizontal column for different time steps. Line elements has been used with time and space discretization $\delta t = 0.5s$ and $\delta x = 0.05m$ respectively.

References

[1] I. Ben Gharbia, J. Jaffré, Gas phase appearance and disappearance as a problem with complementarity constraints, Mathematics and Computers in Simulation 99 (2014) 28–36.
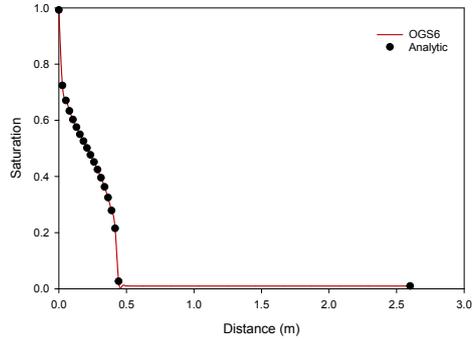
Figure 3: Water saturation, S_w profile of the present result along with analytical solution based on one by McWhorter.

Property	Symbol	Value	Unit
Column length	L	m	2.6
wetting dynamic viscosity	μ_w	$Pa.s$	1.0×10^{-3}
non-wetting dynamic viscosity	μ_{nw}	$Pa.s$	1.0×10^{-3}
wetting phase density	ρ_w	$kg.m^{-3}$	1.0×10^3
Non-wetting phase density	ρ_{nw}	$kg.m^{-3}$	1.0×10^3
Permeability	$\mathbf{K}$	m^2	1.0×10^{-10}
Porosity	n	$--$	3.0×10^{-1}
Residual saturation of water	S_{rw}	$--$	0
Residual saturation of oil	S_{nrw}	$--$	0
Entry pressure	p_d	Pa	5.0×10^3
Soil distribution index	λ	$--$	2.0
Capillary pressure	$p^c(S_{eff})$	Pa	Brooks-Corey model
Relative permeability	$\kappa_{rel}(S_{eff})$	$--$	Brooks-Corey model

Table 2: Material parameters for the McWhorter problem.

[2] W. Gautschi, Repeated integrals of the coerror function, ACM Trans. Math. Software 3 (1977) 301–302.
[3] A. C. Liakopoulos, Variation of the permeability tensor ellipsoid in homogeneous anisotropic soils, Water Resources Research 1 (1) (1965) 135–141.
[4] R. W. Lewis, B. A. Schrefler, The finite element method in the static and dynamic deformation and consolidation of porous media, Vol. 2, Wiley Chichester, 1998.
[5] D. B. McWhorter, D. K. Sunada, Exact integral solutions for two-phase flow, Water Resources Research 26 (3) (1990) 399–413.