One-sided incompressibility constraint for fracture models

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We're concerned here with the fracture closure treatment in our enriched FE implementation (LIE) governed by a fracture normal stiffness K_n:

$$\sigma_\mathrm{n} = K_\mathrm{n} [\![\mathbf{u}]\!] \cdot \mathbf{n}_\mathrm{F} \tag{1}$$

This normal stiffness acts as a proportionality factor between the normal stress transmitted across the fracture and the displacement jump across the fracture, i.e. the relative normal displacement of the two opposing fracture surfaces.

Given a sufficiently high normal traction σ_n, the displacement jump $w_\mathrm{n} = [\![\mathbf{u}]\!] \cdot \mathbf{n}_\mathrm{F}$ will increase (in absolute value) linearly given a constant normal stiffness. This will lead to cases where the current aperture given by

$$b = b_0 + w_n \tag{2}$$

attains physically inadmissible values of $b < 0$, i.e. negative fracture apertures.

Several remedies exist for this case (Belytschko, Moës, et al., 2001; Belytschko, Gracie, et al., 2009) that can involve the introduction of Lagrange multipliers as additional unknowns or Dirichlet-like modifications of the linear system. Here, we choose a simple approach motivated by the constitutive treatment of one-sided incompressible materials (Ehlers et al., 1999). In essence, the normal stiffness is modified by a factor that is one in the initial state ($b = b_0$) and increases towards infinity as $b \to 0$, thus making the fracture increasingly incompressible in the normal direction. Specifically we chose the formulation:

$$K_\mathrm{n} = K_\mathrm{n}^0 \left[1 + \ln^2 \left(\frac{b}{b_0} \right) \right] \tag{3}$$

As intended, $K_\mathrm{n}(b = b_0) = K_\mathrm{n}^0$, where K_n^0 is the initial normal stiffness specified in the project file. Furthermore,

$$\lim_{b \to 0} K_\mathrm{n} = \infty \tag{4}$$

In other words, compression is increasingly penalized as the fracture aperture approaches zero. This formulation is only activated in compression, strictly convex, its derivative is continuous with a value of one at the origin ($w_\mathrm{n} = 0$, or equivalently $b = b_0$) and it is controlled by the keyword `penalty_aperture_cutoff` in the fracture constitutive model section of the project file. The value b^cutoff assigned to this keyword ensures the non-negativity of the argument assigned to the logarithm should the iterative scheme yield a negative aperture. It enters the constitutive equations where we made the assumption that stiffness increases linearly, not logarithmically, if the aperture is below this cutoff value:

$$K_\mathrm{n}^\mathrm{lin} = K_\mathrm{n}|_{b^\mathrm{cutoff}} + \left.\frac{\partial K_\mathrm{n}}{\partial b}\right|_{b^\mathrm{cutoff}} (b - b^\mathrm{cutoff}) \tag{5}$$

The transition between Eq. (3) and Eq. (5) is smooth.

Note that invoking this formulation turns Eq. (1) into a quasi-linear formulation. The additional constitutive non-linearity has to be accounted for by, e.g., allowing more non-linear iterations.

If $b^\mathrm{cutoff} = b_0$, then a standard model with $K_\mathrm{n} = K_\mathrm{n}^0$ is invoked.

In summary:

$$K_\mathrm{n} = \begin{cases} K_\mathrm{n}^0 \left[1 + \ln^2\left(\dfrac{b}{b_0}\right)\right] & b^\mathrm{cutoff} < b < b_0 \\ K_\mathrm{n}^0 \left[1 + \ln^2\left(\dfrac{b^\mathrm{cutoff}}{b_0}\right) + \dfrac{2\ln\left(\frac{b^\mathrm{cutoff}}{b_0}\right)}{b^\mathrm{cutoff}}(b - b^\mathrm{cutoff})\right] & b \leq b^\mathrm{cutoff} \end{cases} \tag{6}$$

References

Belytschko, Ted, Nicolas Moës, et al. (2001). “Arbitrary discontinuities in finite elements”. In: *International Journal for Numerical Methods in Engineering* 50.4, pp. 993–1013.

Belytschko, Ted, Robert Gracie, and Giulio Ventura (2009). “A review of extended/generalized finite element methods for material modeling”. In: *Modelling and Simulation in Materials Science and Engineering* 17.4, p. 043001.

Ehlers, W. and G. Eipper (1999). “Finite Elastic Deformations in Liquid-Saturated and Empty Porous Solids”. In: *Transport in Porous Media* 34.1, pp. 179–191.