F bar method for total Lagrangian formulation

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July 3, 2024

According to the references [1, 2], a modified deformation gradient, $\bar{\mathbf{F}}$, is introduced to compute stresses in order to alleviate the spurious locking exhibited by the standard bi-linear and tri-linear elements near the incompressible limit. The deformation gradient $\mathbf{F}$ can be expressed as a composition of dilatational change and deviatoric change as

$$\mathbf{F} = \mathbf{F}_d \mathbf{F}_v$$

With $(\mathbf{F}_0)_v = \det[\mathbf{F}_0]^{1/\mathrm{n}}\mathbf{I}$ and $\mathbf{F}_d = \det[\mathbf{F}]^{-1/\mathrm{n}}\mathbf{F}_d$, $\bar{\mathbf{F}}$ is defined as the composition of the deviatoric component of $\mathbf{F}$ with the volumetric component of $\mathbf{F}_0$ as

$$\bar{\mathbf{F}} := \mathbf{F}_d (\mathbf{F}_0)_v = \left(\frac{\det[\mathbf{F}_0]}{\det[\mathbf{F}]} \right)^{\frac{1}{\mathrm{n}}} \mathbf{F}$$

with $\mathbf{F}_0$ the value of $\mathbf{F}$ at the element center, n the space dimension.

Alternatively, $\mathbf{F}_0$ can be computed as a average value as

$$\mathbf{F}_0 = \frac{\int_{\Omega_e} \mathbf{F} \mathrm{d}\Omega}{\int_{\Omega_e} \mathrm{d}\Omega}$$

Hereafter, we denote $\left(\frac{\det[\mathbf{F}_0]}{\det[\mathbf{F}]} \right)^{\frac{1}{\mathrm{n}}}$ as α.

1 Equilibrium equations

We assume that $\{\Omega^t : \mathbf{x} \in \mathbb{R}^n\}$ is the current deformed configuration, $\{\Omega : \mathbf{X} \in \mathbb{R}^n\}$ is the reference configuration, and $\phi(\mathbf{X})$ is the coordinate mapping $\{\phi(\mathbf{X}) : \Omega \to \Omega^t\}$ such as $\mathbf{x} = \phi(\mathbf{X})$. The displacement and its gradient can be written as

$$\mathbf{u} = \mathbf{x} - \mathbf{X} = \phi(\mathbf{X}) - \mathbf{X}, \tag{1}$$
$$\mathbf{F} = \nabla \phi(\mathbf{X}). \tag{2}$$

Let $\mathcal{S}$ is the space of admissible deformations defined by

$$\mathcal{S} = \phi : \Omega \to \mathbb{R}^n | \det(\nabla_X \phi) > 0, \phi|_{\partial\Omega_\phi} = \phi_b, \tag{3}$$

and $\mathcal{V}_\phi$ is the tangent space to $\mathcal{S}$ at ϕ as

$$\mathcal{V}_\phi = d\mathbf{w} \circ \phi : \Omega \to \mathbb{R}^n | \det(\nabla_X \phi) > 0, d\mathbf{w} \circ \phi|_{\partial\Omega_\phi} = 0, \tag{4}$$

In the total Lagrangian formulation, the equilibrium equations are derived from the principle of virtual work in the reference configuration Ω. This leads to find $\phi \in \mathcal{S}$ such that $\forall d\mathbf{w} \in \mathcal{V}_\phi$ it satisfies

$$\int_{\Omega_e} \mathbf{S} : d\bar{\mathbf{E}}(d\mathbf{w}) \mathrm{d}\Omega = \int_{\Omega_e} \mathbf{f} \cdot d\mathbf{w} \mathrm{d}\Omega + \int_{\partial\Omega|_\tau} d\mathbf{w} \cdot \tau \mathrm{d}\Gamma \tag{5}$$

where $\mathbf{S}$ is the second Piola-Kirchhoff stress tensor, $\bar{\mathbf{E}}$ is the Green-Lagrange strain calculated with $\bar{\mathbf{F}}$ and $\mathbf{f}$ is the external load vector, The Green-Lagrange strain is defined by

$$\mathbf{E} = \frac{1}{2}(\mathbf{F}^{\mathrm{T}}\mathbf{F} - \mathbf{I}), \tag{6}$$

Using the F bar method, the modified Green-Lagrange strain $\bar{\mathbf{E}}$ is

$$\bar{\mathbf{E}} = \frac{1}{2}(\bar{\mathbf{F}}^{\mathrm{T}}\bar{\mathbf{F}} - \mathbf{I}), \tag{7}$$

$$= \frac{1}{2}(\alpha^2\mathbf{F}^{\mathrm{T}}\mathbf{F} - \alpha^2\mathbf{I} + \alpha^2\mathbf{I} - \mathbf{I}), \tag{8}$$

$$= \alpha^2\mathbf{E} + \frac{1}{2}(\alpha^2 - 1)\mathbf{I}. \tag{9}$$

Consequently, the stress is computed as

$$\mathbf{S} := \mathbf{S}(\bar{\mathbf{E}}). \tag{10}$$

2 Linearization

2.1 Basic derivatives

To linearize the equilibrium equations, we first derive some fundamental derivatives.

2.1.1 Directional derivatives

The directional derivative of a multivariable differentiable (scalar) function along a given vector $\mathbf{v}$ at a given point $\mathbf{x}$ intuitively represents the instantaneous rate of change of the function, moving through $\mathbf{x}$ with a velocity specified by $\mathbf{v}$: The directional derivative of a scalar function $f(\mathbf{x}), \mathbf{x} \in \mathbb{R}^n$ along a vector $\mathbf{v} \in \mathbb{R}^n$ is the function $\nabla_{\mathbf{v}} f(\mathbf{x})$ defined by the limit:

$$\nabla_{\mathbf{v}} f(\mathbf{x}) = \lim_{h\to 0} \frac{f(\mathbf{x} + h\mathbf{v}) - f(\mathbf{x})}{h} = \frac{\partial}{\partial h} f(\mathbf{x} + h\mathbf{v}) \mid_{\lim_{h\to 0}} \tag{11}$$

If $f(\mathbf{x})$ is differentiable at $\mathbf{x}$, the following equation holds after applying the first order Taylor approximation to $f(\mathbf{x} + h\mathbf{v})$ in the above definition

$$\nabla_{\mathbf{v}} f(\mathbf{x}) = \nabla f(\mathbf{x})\mathbf{v} = \frac{\partial}{\partial h} f(\mathbf{x} + h\mathbf{v}) \mid_{\lim_{h\to 0}} \tag{12}$$

We will use the definition of directional derivatives to simplify the linearization.

2.1.2 Virtual deformation gradient $d\mathbf{F}$

$$d\mathbf{F} = \nabla_\phi \mathbf{F} d\mathbf{w} = \frac{\partial}{\partial h}\mathbf{F}(\phi + h d\mathbf{w}) \mid_{\lim_{h\to 0}} = \nabla d\mathbf{w} \tag{13}$$

2.1.3 Virtual strain $d\mathbf{E}$

$$d\mathbf{E} = \nabla_\phi \mathbf{E} d\mathbf{w} = \frac{1}{2}((\nabla d\mathbf{w})^{\mathrm{T}}\mathbf{F} + \mathbf{F}^{\mathrm{T}}\nabla d\mathbf{w}) \tag{14}$$

2.1.4 Virtual strain $d\bar{\mathbf{E}}$

Therefore, the variation of the modified Green-Lagrange strain gives

$$d\bar{\mathbf{E}} = \alpha^2 d\mathbf{E} + \alpha(2\mathbf{E} + \mathbf{I})d\alpha. \tag{15}$$

Note that the derivative of the determinant of a matrix with respect to the matrix itself is used to obtain the above derivatives, which is

$$\frac{\partial}{\partial \mathbf{A}}(\det(\mathbf{A})) = \det(\mathbf{A})\mathbf{A}^{-\mathrm{T}}. \tag{16}$$

This gives

$$d\alpha = \frac{\alpha}{\mathrm{n}}(\mathbf{F}_0^{-\mathrm{T}} : d\mathbf{F}_0 - \mathbf{F}^{-\mathrm{T}} : d\mathbf{F}), \tag{17}$$

$$= \frac{\alpha}{\mathrm{n}}\left(\mathbf{F}_0^{-\mathrm{T}} : \nabla d\mathbf{w}_0 - \mathbf{F}^{-\mathrm{T}} : \nabla d\mathbf{w}\right), \tag{18}$$

$$\tag{19}$$

Consequently,

$$d\bar{\mathbf{E}} = \alpha^2 d\mathbf{E} + \alpha(2\mathbf{E} + \mathbf{I})\left(\frac{\alpha}{\mathrm{n}}\left(\mathbf{F}_0^{-\mathrm{T}} : \nabla d\mathbf{w}_0 - \mathbf{F}^{-\mathrm{T}} : \nabla d\mathbf{w}\right)\right) \tag{20}$$

$$= \alpha^2\left(d\mathbf{E} + \frac{1}{\mathrm{n}}(2\mathbf{E} + \mathbf{I})\left(\mathbf{F}_0^{-\mathrm{T}} : \nabla d\mathbf{w}_0 - \mathbf{F}^{-\mathrm{T}} : \nabla d\mathbf{w}\right)\right) \tag{21}$$

2.1.5 Jacobian

Assume that the body force $\mathbf{f}$ and the traction τ are independent of the displacement, the Jacobian for the Newton-Raphson method can be obtained by deriving the variation of the virtual strain energy as

$$d_u \int_{\Omega_e} \mathbf{S}(\bar{\mathbf{E}}) : d\bar{\mathbf{E}} \mathrm{d}\Omega = \int_{\Omega_e} \nabla_u(\mathbf{S}(\bar{\mathbf{E}}) : d\bar{\mathbf{E}})\delta\mathbf{u}\mathrm{d}\Omega. \tag{22}$$

with $\delta\mathbf{u} \in \mathcal{V}_\phi$

According to the directional derivative rule,

$$\nabla_u(\mathbf{S}(\bar{\mathbf{E}}) : d\bar{\mathbf{E}})\delta\mathbf{u} = \frac{\partial}{\partial h}\left(\mathbf{S}(\bar{\mathbf{E}}(\phi + h\delta\mathbf{u})) : d\bar{\mathbf{E}}(\phi + h\delta\mathbf{u})\right)|_{\lim_{h\to 0}}, \tag{23}$$

$$= \frac{\partial \mathbf{S}(\bar{\mathbf{E}}(\phi + h\delta\mathbf{u})}{\partial \bar{\mathbf{E}}(\phi + h\delta\mathbf{u})} : \frac{\partial \bar{\mathbf{E}}(\phi + h\delta\mathbf{u})}{\partial h} : d\bar{\mathbf{E}}(\phi + h\delta\mathbf{u})|_{\lim_{h\to 0}} \tag{24}$$

$$+ \mathbf{S}(\bar{\mathbf{E}}(\phi + h\delta\mathbf{u})) : \frac{\partial}{\partial h}(d\bar{\mathbf{E}}(\phi + h\delta\mathbf{u}))|_{\lim_{h\to 0}}, \tag{25}$$

where $\partial \mathbf{S}(\bar{\mathbf{E}}(\phi + h\delta\mathbf{u})/\partial \bar{\mathbf{E}}(\phi + h\delta\mathbf{u}) \mid_{\lim_{h\to 0}} = \partial \mathbf{S}(\bar{\mathbf{E}}(\phi)/\partial \bar{\mathbf{E}}(\phi)$ is the material tangential, which is a forth order tensor, hereafter we denote it as $\mathbf{C}$.

Since

$$\frac{\partial \bar{\mathbf{E}}(\phi + h\delta\mathbf{u})}{\partial h} \mid_{\lim_{h\to 0}} = \nabla_\phi \bar{\mathbf{E}}(\phi)\delta\mathbf{u} = \delta\bar{\mathbf{E}}(\phi), \tag{26}$$

we have

$$\delta\bar{\mathbf{E}} = \alpha^2 \left(\delta\mathbf{E} + \frac{1}{\mathrm{n}}(2\mathbf{E} + \mathbf{I}) \left(\mathbf{F}_0^{-\mathrm{T}} : \nabla\delta\mathbf{u}_0 - \mathbf{F}^{-\mathrm{T}} : \nabla\delta\mathbf{u} \right) \right), \tag{27}$$

by the same way deriving $d\bar{\mathbf{E}}$.

Expanding $\partial(d\bar{\mathbf{E}}(\phi + h\delta\mathbf{u})/\partial h \mid_{\lim_{h\to 0}}$ leads to

$$\frac{\partial}{\partial h}(d\bar{\mathbf{E}}(\phi + h\delta\mathbf{u})) \mid_{\lim_{h\to 0}} = \frac{\partial}{\partial h} \left(\alpha^2 \left(d\mathbf{E} + \frac{1}{\mathrm{n}}(2\mathbf{E} + \mathbf{I}) \left(\mathbf{F}_0^{-\mathrm{T}} : \nabla d\mathbf{w}_0 - \mathbf{F}^{-\mathrm{T}} : \nabla d\mathbf{w} \right) \right) \right) \mid_{\lim_{h\to 0}} \tag{28}$$

$$= 2\alpha \left(d\mathbf{E} + \frac{1}{\mathrm{n}}(2\mathbf{E} + \mathbf{I}) \left(\mathbf{F}_0^{-\mathrm{T}} : \nabla d\mathbf{w}_0 - \mathbf{F}^{-\mathrm{T}} : \nabla d\mathbf{w} \right) \right) \frac{\partial \alpha}{\partial h} \mid_{\lim_{h\to 0}} \tag{29}$$

$$+ \alpha^2 \frac{\partial d\mathbf{E}}{\partial h} \mid_{\lim_{h\to 0}} + \frac{2\alpha^2}{\mathrm{n}} \frac{\partial \mathbf{E}}{\partial h} \mid_{\lim_{h\to 0}} \left(\mathbf{F}_0^{-\mathrm{T}} : \nabla d\mathbf{w}_0 - \mathbf{F}^{-\mathrm{T}} : \nabla d\mathbf{w} \right) \tag{30}$$

$$+ \frac{\alpha^2}{\mathrm{n}}(2\mathbf{E} + \mathbf{I}) \left(\frac{\partial}{\partial h}(\mathbf{F}_0^{-\mathrm{T}}) \mid_{\lim_{h\to 0}} : \nabla d\mathbf{w}_0 - \frac{\partial}{\partial h}(\mathbf{F}^{-\mathrm{T}}) \mid_{\lim_{h\to 0}} : \nabla d\mathbf{w} \right) \tag{31}$$

where

$$\frac{\partial \alpha}{\partial h} \mid_{\lim_{h\to 0}} = \frac{\partial \alpha(\phi + h\delta\mathbf{u})}{\partial h} \mid_{\lim_{h\to 0}} = \nabla_{\mathbf{u}} \alpha \delta\mathbf{u} = \delta\alpha = \frac{\alpha}{\mathrm{n}} \left(\mathbf{F}_0^{-\mathrm{T}} : \nabla\delta\mathbf{u}_0 - \mathbf{F}^{-\mathrm{T}} : \nabla\delta\mathbf{u} \right), \tag{32}$$

$$\frac{\partial d\mathbf{E}}{\partial h} \mid_{\lim_{h\to 0}} = \frac{1}{2}\frac{\partial}{\partial h}((\nabla d\mathbf{w})^{\mathrm{T}}\mathbf{F} + \mathbf{F}^{\mathrm{T}}\nabla d\mathbf{w}) \mid_{\lim_{h\to 0}} \tag{33}$$

$$= \frac{1}{2}((\nabla d\mathbf{w})^{\mathrm{T}}\nabla\delta\mathbf{u} + (\nabla\delta\mathbf{u})^{tr}\nabla d\mathbf{w}), \tag{34}$$

$$\frac{\partial \mathbf{E}}{\partial h} \mid_{\lim_{h\to 0}} = \frac{\partial \mathbf{E}(\phi + h\delta\mathbf{u})}{\partial h} \mid_{\lim_{h\to 0}} = \frac{1}{2}((\nabla\delta\mathbf{u})^{\mathrm{T}}\mathbf{F} + \mathbf{F}^{\mathrm{T}}\nabla\delta\mathbf{u}), \tag{35}$$

$$\frac{\partial \mathbf{F}_0^{-\mathrm{T}}}{\partial h} \mid_{\lim_{h\to 0}} = \frac{\partial F_0^{-\mathrm{T}}}{\partial \mathbf{F}_0} \frac{\partial \mathbf{F}_0(\phi + h\delta\mathbf{u})}{\partial h} \mid_{\lim_{h\to 0}} = -\mathbf{F}_0^{-\mathrm{T}} \otimes \mathbf{F}_0^{-\mathrm{T}} : \nabla\delta\mathbf{u}_0, \tag{36}$$

$$\frac{\partial \mathbf{F}^{-\mathrm{T}}}{\partial h} \mid_{\lim_{h\to 0}} = \frac{\partial F^{-\mathrm{T}}}{\partial \mathbf{F}} \frac{\partial \mathbf{F}(\phi + h\delta\mathbf{u})}{\partial h} \mid_{\lim_{h\to 0}} = -\mathbf{F}^{-\mathrm{T}} \otimes \mathbf{F}^{-\mathrm{T}} : \nabla\delta\mathbf{u}. \tag{37}$$

We have

$$\frac{\partial}{\partial h}(d\bar{\mathbf{E}}(\phi + h\delta\mathbf{u})) \mid_{\lim_{h\to 0}} = \delta(d\bar{\mathbf{E}}) \tag{38}$$

$$= \frac{2\alpha^2}{\mathrm{n}}\left(d\mathbf{E} + \frac{1}{\mathrm{n}}(2\mathbf{E}+\mathbf{I})\left(\mathbf{F}_0^{-\mathrm{T}} : \nabla d\mathbf{w}_0 - \mathbf{F}^{-\mathrm{T}} : \nabla d\mathbf{w}\right)\right) \tag{39}$$

$$\left(\mathbf{F}_0^{-\mathrm{T}} : \nabla\delta\mathbf{u}_0 - \mathbf{F}^{-\mathrm{T}} : \nabla\delta\mathbf{u}\right) \tag{40}$$

$$+ \alpha^2\frac{\partial d\mathbf{E}}{\partial h} \mid_{\lim_{h\to 0}} \tag{41}$$

$$+ \frac{\alpha^2}{\mathrm{n}}((\nabla\delta\mathbf{u})^{\mathrm{T}}\mathbf{F} + \mathbf{F}^{\mathrm{T}}\nabla\delta\mathbf{u})\left(\mathbf{F}_0^{-\mathrm{T}} : \nabla d\mathbf{w}_0 - \mathbf{F}^{-\mathrm{T}} : \nabla d\mathbf{w}\right) \tag{42}$$

$$- \frac{\alpha^2}{\mathrm{n}}(2\mathbf{E}+\mathbf{I})\left(\mathbf{F}_0^{-\mathrm{T}} \otimes \mathbf{F}_0^{-\mathrm{T}} : \nabla\delta\mathbf{u}_0 : \nabla d\mathbf{w}_0 - \mathbf{F}^{-\mathrm{T}} \otimes \mathbf{F}^{-\mathrm{T}} : \nabla\delta\mathbf{u} : \nabla d\mathbf{w}\right) \tag{43}$$

Finally, we obtain the expression the variation of the virtual strain energy as

$$\int_{\Omega_e} \nabla_u(\mathbf{S}(\bar{\mathbf{E}}) : d\bar{\mathbf{E}})\delta\mathbf{u}\mathrm{d}\Omega = \int_{\Omega_e} \mathbf{C} : \delta\bar{\mathbf{E}} : d\bar{\mathbf{E}}\mathrm{d}\Omega \tag{44}$$

$$+ \int_{\Omega_e} \mathbf{S}(\bar{\mathbf{E}}) : \delta(d\bar{\mathbf{E}}(\phi + h\delta\mathbf{u}))\mathrm{d}\Omega. \tag{45}$$

3 Finite element

After the discretization with the Galerkin approach, we have

$$\delta\mathbf{u} = \mathbf{N}\hat{\delta\mathbf{u}}, \quad d\mathbf{u} = \mathbf{N}\hat{d\mathbf{u}} \tag{46}$$

with $\mathbf{N}$ the shape functions, $\hat{\delta\mathbf{u}}$ and $\hat{d\mathbf{u}}$ the arbitrary virtual nodal displacements. This gives

$$d\mathbf{E} = \frac{\partial\mathbf{E}}{\partial\hat{\mathbf{u}}} : \hat{d\mathbf{u}}, \quad \delta\mathbf{E} = \frac{\partial\mathbf{E}}{\partial\hat{\mathbf{u}}} : \hat{\delta\mathbf{u}}. \tag{47}$$

Assuming that stress tensor and strain tensor are symmetry, and considering the matrix form of $\partial\mathbf{E}/\partial\hat{\mathbf{u}}$ gives

$$\frac{\partial\mathbf{E}}{\partial\hat{\mathbf{u}}} := \mathrm{B}. \tag{48}$$

where $\mathrm{B} = \mathrm{B}_l + \mathrm{B}_{nl}$ with B_n and B_{nl} the linear and non-linear parts of the B matrix, respectively.

3.1 Modified Jacobian

Additional B matrix from $d\bar{\mathbf{E}}$ and $\delta\bar{\mathbf{E}}$

As for $d\bar{\mathbf{E}}$, let's denote $\alpha^2(2\mathbf{E}+\mathbf{I})\left(\mathbf{F}_0^{-\mathrm{T}} : \nabla dw_0 - \mathbf{F}^{-\mathrm{T}} : \nabla dw\right)/\mathrm{n}$ as $d\bar{\varepsilon}$, which gives

$$d\bar{\mathbf{E}} = \alpha^2 d\mathbf{E} + d\bar{\varepsilon} \tag{49}$$

Expanding $\left(\mathbf{F}_0^{-\mathrm{T}} : \nabla d\mathbf{w}_0 - \mathbf{F}^{-\mathrm{T}} : \nabla d\mathbf{w}\right)$ gives:

$$\left(\mathbf{F}_0^{-\mathrm{T}} : \nabla d\mathbf{w}_0 - \mathbf{F}^{-\mathrm{T}} : \nabla d\mathbf{w}\right) = (\mathbf{F}_0^{-T})_{ij} d\hat{w}_i^L N_{,j}^L(\xi_0) - (\mathbf{F}^{-T})_{ij} d\hat{w}_i^L N_{,j}^L = (\mathbf{q}_0 - \mathbf{q}) d\hat{\mathbf{w}}, \quad (50)$$

where $\mathbf{q}$ is a $1 \times \mathrm{n} \cdot \mathrm{NE}$ matrix or a transposed vector as

$$\mathbf{q} = (\mathbf{q}_{\mathrm{col}\ 1}, \mathbf{q}_{\mathrm{col}\ 2}), \text{or } \mathbf{q} = (\mathbf{q}_{\mathrm{col}\ 1}, \mathbf{q}_{\mathrm{col}\ 2}, \mathbf{q}_{\mathrm{col}\ 3}) \quad (51)$$

$$\mathbf{q}_{\mathrm{col}\ \mathrm{i}} = \left((\mathbf{F}^{-1})_{ji} N_{,j}^1, \quad (\mathbf{F}^{-1})_{ji} N_{,j}^2, \quad \cdots, (\mathbf{F}^{-1})_{ji} N_{,j}^{\mathrm{NE}}\right), \quad i, j = 1, \cdots, \dim. \quad (52)$$

with NE the number of nodes, and $\mathbf{q}_0$ the value of $\mathbf{q}$ at the element center ξ_0.

This can be written into a matrix form as

$$[d\bar{\varepsilon}] = \hat{\mathrm{B}} d\hat{\mathbf{w}}, \quad (53)$$

with $[d\bar{\varepsilon}]$ the vector form of $d\bar{\varepsilon}$, and

$$\hat{\mathrm{B}} = \frac{\alpha^2}{\mathrm{n}} [2\mathbf{E} + \mathbf{I}](\mathbf{q}_0 - \mathbf{q}), \quad (54)$$

where $[2\mathbf{E} + \mathbf{I}] = (2E_{11} + 1, 2E_{22} + 1, 2E_{33} + 1, 2E_{12})^{\mathrm{T}}$ for plane strain problems, and $[2\mathbf{E} + \mathbf{I}] = (2E_{11} + 1, 2E_{22} + 1, 2E_{33} + 1, 2E_{12}, 2E_{23}, 2E_{13})^{\mathrm{T}}$ for 3D problems, respectively.

Note: the shear strain E_{12}, E_{23} , E_{13} are assumed being scaled with $\sqrt{2}$ for the computation with the Kelvin vector.

The same for $\delta\bar{\mathbf{E}}$, we have $\delta\bar{\varepsilon} = \hat{\mathrm{B}} \delta\hat{\mathbf{u}}$. Since $d\bar{\mathbf{E}} = \alpha^2 d\mathbf{E} + d\bar{\varepsilon}$,

$$[d\bar{\mathbf{E}}] = (\alpha^2 \mathrm{B} + \hat{\mathrm{B}}) d\hat{\mathbf{w}} \quad (55)$$

where $[d\bar{\mathbf{E}}]$ means the vector form of $d\bar{\mathbf{E}}$.

We denote $\alpha^2 \mathrm{B} + \hat{\mathrm{B}}$ as $\bar{\mathrm{B}}$, which simplifies the expression of the Jacobian from $\int_{\Omega_e} \mathbf{C} : \delta\bar{\mathbf{E}} : d\bar{\mathbf{E}} \mathrm{d}\Omega$ as

$$\int_{\Omega_e} \mathbf{C} : (\delta\bar{\mathbf{E}}) : d\bar{\mathbf{E}} \mathrm{d}\Omega \xrightarrow{\text{matrix-vector form}} \int_{\Omega_e} \bar{\mathrm{B}}^{\mathrm{T}} [\mathbf{C}] \bar{\mathrm{B}} \mathrm{d}\Omega \quad (56)$$

where $[\mathbf{C}]$ is matrix from of $\mathbf{C}$.

3.2 Additional contributions to Jacobian from $\int_{\Omega_e} \delta(d\bar{\mathbf{E}}) \mathrm{d}\Omega$

3.2.1 Term $\int_{\Omega_e} \alpha^2 \mathbf{S} : \delta(d\mathbf{E}) \mathrm{d}\Omega = \int_{\Omega_e} \mathbf{S} : \alpha^2 \partial d\mathbf{E} / \partial h \mathrm{d}\Omega \mid_{\lim_{h \to 0}}$

From this term, we obtain the standard G matrix related Jacobian contribution as

$$\int_{\Omega_e} \alpha^2 \mathbf{G}^{\mathrm{T}} [[\mathbf{S}]] \mathbf{G} \mathrm{d}\Omega \quad (57)$$

with $[[\mathbf{S}]]$ for a matrix with stress matrix as diagonal blocks.

3.2.2 Term with $\frac{2\alpha^2}{\mathbf{n}} \left(d\mathbf{E} + \frac{1}{\mathbf{n}}(2\mathbf{E} + \mathbf{I}) \left(\mathbf{F}_0^{-\mathbf{T}} : \nabla d\mathbf{w}_0 - \mathbf{F}^{-\mathbf{T}} : \nabla d\mathbf{w}\right)\right)$ $\left(\mathbf{F}_0^{-\mathbf{T}} : \nabla \delta\mathbf{u}_0 - \mathbf{F}^{-\mathbf{T}} : \nabla \delta\mathbf{u}\right)$

The corresponding term in the linearized weak form is

$$\int_{\Omega_e} \mathbf{S} : \frac{2\alpha^2}{\mathrm{n}} \left(d\mathbf{E} + \frac{1}{\mathrm{n}}(2\mathbf{E} + \mathbf{I}) \left(\mathbf{F}_0^{-\mathrm{T}} : \nabla d\mathbf{w}_0 - \mathbf{F}^{-\mathrm{T}} : \nabla d\mathbf{w}\right)\right) \quad (58)$$

$$\left(\mathbf{F}_0^{-\mathrm{T}} : \nabla \delta\mathbf{u}_0 - \mathbf{F}^{-\mathrm{T}} : \nabla \delta\mathbf{u}\right) \mathrm{d}\Omega, \quad (59)$$

which can be written:

$$\frac{2}{\mathrm{n}}\int_{\Omega_e}\mathbf{S}:d\bar{\mathbf{E}}\left(\mathbf{F}_0^{-\mathrm{T}}:\nabla\delta\mathbf{u}_0-\mathbf{F}^{-\mathrm{T}}:\nabla\delta\mathbf{u}\right)\mathrm{d}\Omega, \tag{60}$$

Note that

$$\mathbf{S}:d\bar{\mathbf{E}}=(\bar{\mathrm{B}}d\hat{\mathbf{w}})^{\mathrm{T}}[\mathbf{S}] \tag{61}$$

with $[\mathbf{S}]$ the stress in vector type, e.g. the stress in the Kevlin vector.

While

$$\left(\mathbf{F}_0^{-\mathrm{T}}:\nabla\delta\mathbf{u}_0-\mathbf{F}^{-\mathrm{T}}:\nabla\delta\mathbf{u}\right)=(\mathbf{q}_0-\mathbf{q})\hat{\delta\mathbf{u}} \tag{62}$$

Therefore the additional Jocobian obtained from this term is

$$\frac{2}{\mathrm{n}}\int_{\Omega_e}(\bar{\mathrm{B}})^{\mathrm{T}}[\mathbf{S}](\mathbf{q}_0-\mathbf{q})\mathrm{d}\Omega \tag{63}$$

3.2.3 Term with $\frac{2\alpha^2}{\mathbf{n}}\frac{\partial\mathbf{E}}{\partial h}|_{\lim_{h\to 0}}\left(\mathbf{F}_0^{-\mathbf{T}}:\nabla d\mathbf{w}_0-\mathbf{F}^{-\mathbf{T}}:\nabla d\mathbf{w}\right)$

Note that $\frac{\partial\mathbf{E}}{\partial h}|_{\lim_{h\to 0}}=\delta\mathbf{E}$ in that term, the term corresponding integration term is

$$\int_{\Omega_e}\frac{2\alpha^2}{\mathrm{n}}\mathbf{S}:\delta\mathbf{E}\left(\mathbf{F}_0^{-\mathrm{T}}:\nabla d\mathbf{w}_0-\mathbf{F}^{-\mathrm{T}}:\nabla d\mathbf{w}\right)\mathrm{d}\Omega \tag{64}$$

We see that

$$\mathbf{S}:\delta\mathbf{E}=[\mathbf{S}]^{tr}\mathrm{B}\hat{\delta\mathbf{u}} \tag{65}$$

with $[\mathbf{S}]$ the stress in vector type.

The same for $\left(\mathbf{F}_0^{-\mathrm{T}}:\nabla\delta\mathbf{u}_0-\mathbf{F}^{-\mathrm{T}}:\nabla\delta\mathbf{u}\right)$, the discretized of $\left(\mathbf{F}_0^{-\mathrm{T}}:\nabla d\mathbf{w}_0-\mathbf{F}^{-\mathrm{T}}:\nabla d\mathbf{w}\right)$ takes the form

$$\left(\mathbf{F}_0^{-\mathrm{T}}:\nabla d\mathbf{w}_0-\mathbf{F}^{-\mathrm{T}}:\nabla d\mathbf{w}\right):=(\mathbf{q}_0-\mathbf{q})\hat{d\mathbf{w}}=\hat{d\mathbf{w}}^{tr}(\mathbf{q}_0^{tr}-\mathbf{q}^{tr}) \tag{66}$$

Therefore the integration can be written as

$$\frac{2}{\mathrm{n}}\int_{\Omega_e}\hat{d\mathbf{w}}^{tr}\alpha^2(\mathbf{q}_0^{tr}-\mathbf{q}^{tr})[\mathbf{S}]^{tr}\mathrm{B}\hat{\delta\mathbf{u}}\mathrm{d}\Omega \tag{67}$$

This Jacobian contribution from this integration is

$$\frac{2}{\mathrm{n}}\int_{\Omega_e}\alpha^2(\mathbf{q}_0^{tr}-\mathbf{q}^{tr})[\mathbf{S}]^{tr}\mathrm{B}\mathrm{d}\Omega \tag{68}$$

3.2.4 Term with $-\frac{\alpha^2}{\mathbf{n}}(2\mathbf{E}+\mathbf{I})\left(\mathbf{F}_0^{-\mathbf{T}}\otimes\mathbf{F}_0^{-\mathbf{T}}:\nabla\delta\mathbf{u}_0:\nabla d\mathbf{w}_0-\mathbf{F}^{-\mathbf{T}}\otimes\mathbf{F}^{-\mathbf{T}}:\nabla\delta\mathbf{u}:\nabla d\mathbf{w}\right)$

The corresponding integration is

$$-\frac{1}{\mathrm{n}}\int_{\Omega_e}\alpha^2\mathbf{S}:(2\mathbf{E}+\mathbf{I})\left(\mathbf{F}_0^{-\mathrm{T}}\otimes\mathbf{F}_0^{-\mathrm{T}}:\nabla\delta\mathbf{u}_0:\nabla d\mathbf{w}_0-\mathbf{F}^{-\mathrm{T}}\otimes\mathbf{F}^{-\mathrm{T}}:\nabla\delta\mathbf{u}:\nabla d\mathbf{w}\right)\mathrm{d}\Omega \tag{69}$$

Note that $\mathbf{F}^{-\mathrm{T}}\otimes\mathbf{F}^{-\mathrm{T}}:\nabla\delta\mathbf{u}:\nabla d\mathbf{w}=(\mathbf{F}^{-\mathrm{T}}:d\mathbf{w})(\mathbf{F}^{-\mathrm{T}}:\nabla\delta\mathbf{u})$

From the above description, we know that $\mathbf{F}^{-\mathrm{T}} : \nabla\delta\mathbf{u} := \mathbf{q}\hat{\delta\mathbf{u}}$ and $\mathbf{F}_0^{-\mathrm{T}} : \nabla\delta\mathbf{u} := \mathbf{q}_0\hat{\delta\mathbf{u}}$ after discretization. Therefore, the integration can be written as

$$-\frac{1}{\mathrm{n}}\int_{\Omega_e}\alpha^2\mathbf{S} : (2\mathbf{E}+\mathbf{I})\left(\mathbf{F}_0^{-\mathrm{T}}\otimes\mathbf{F}_0^{-\mathrm{T}} : \nabla\delta\mathbf{u}_0 : \nabla d\mathbf{w}_0 - \mathbf{F}^{-\mathrm{T}}\otimes\mathbf{F}^{-\mathrm{T}} : \nabla\delta\mathbf{u} : \nabla d\mathbf{w}\right)\mathrm{d}\Omega \tag{70}$$

$$= -\frac{1}{\mathrm{n}}\int_{\Omega_e}\alpha^2\mathbf{S} : (2\mathbf{E}+\mathbf{I})\left((\mathbf{q}_0 d\mathbf{w})^{\mathrm{T}}\mathbf{q}_0\delta\mathbf{u} - (\mathbf{q}d\mathbf{w})^{\mathrm{T}}\mathbf{q}\delta\mathbf{u}\right)\mathrm{d}\Omega, \tag{71}$$

$$= -\frac{1}{\mathrm{n}}\int_{\Omega_e}\alpha^2\mathbf{S} : (2\mathbf{E}+\mathbf{I})\left((d\mathbf{w})^{\mathrm{T}}\mathbf{q}_0^{tr}\mathbf{q}_0\delta\mathbf{u} - d\mathbf{w}^{\mathrm{T}}\mathbf{q}^{tr}\mathbf{q}\delta\mathbf{u}\right)\mathrm{d}\Omega, \tag{72}$$

Therefore the Jacobian contribution from this term is

$$-\frac{1}{\mathrm{n}}\int_{\Omega_e}\alpha^2\mathbf{S} : (2\mathbf{E}+\mathbf{I})\left(\mathbf{q}_0^{tr}\mathbf{q}_0 - \mathbf{q}^{tr}\mathbf{q}\right)\mathrm{d}\Omega. \tag{73}$$

Note that $\mathbf{S} : (2\mathbf{E}+\mathbf{I})$ is a scalar, and it can be computed by the dot product of stress vectors as $[\mathbf{S}]\cdot[(2\mathbf{E}+\mathbf{I})]$.

3.3 Jacobian and residual

Finally, we obtain the Jacobian for the total Lagrange formulation with the F bar method:

$$\int_{\Omega_e}\bar{\mathrm{B}}^{\mathrm{T}}[\mathbf{C}]\bar{\mathrm{B}}\mathrm{d}\Omega + \int_{\Omega_e}\alpha^2\mathbf{G}^{\mathrm{T}}[[\mathbf{S}]]\mathbf{G}\mathrm{d}\Omega + \frac{2}{\mathrm{n}}\int_{\Omega_e}(\bar{\mathrm{B}})^{\mathrm{T}}[\mathbf{S}](\mathbf{q}_0-\mathbf{q})\mathrm{d}\Omega \tag{74}$$

$$+\frac{2}{\mathrm{n}}\int_{\Omega_e}\alpha^2(\mathbf{q}_0^{tr}-\mathbf{q}^{tr})[\mathbf{S}]^{tr}\mathrm{B}\mathrm{d}\Omega - \frac{1}{\mathrm{n}}\int_{\Omega_e}\alpha^2\mathbf{S} : (2\mathbf{E}+\mathbf{I})\left(\mathbf{q}_0^{tr}\mathbf{q}_0 - \mathbf{q}^{tr}\mathbf{q}\right)\mathrm{d}\Omega \tag{75}$$

With the equilibrium equation (5), the discretized residual is

$$\mathbf{R} = \int_{\Omega_e}\bar{\mathrm{B}}[\mathbf{S}]\mathrm{d}\Omega - \int_{\Omega_e}\mathbf{f}\mathrm{N}\mathrm{d}\Omega - \int_{\partial\Omega|_\tau}\tau\mathrm{N}\mathrm{d}\Gamma = \mathbf{0} \tag{76}$$

with N the shape function matrix.

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